Lorentzian Model of Spatially Coherent Noise Field in Narrowband Direction Finding

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Abstract: When studying the radiation coming from far field sources using an array of sensors, besides the internal thermal noise, the received wave field is always perturbed by an external noise field, which can be temporally and spatially coherent to some degree, temporally incoherent and spatially coherent, spatially incoherent and temporally correlated or finally, the incoherence in both domains. Thus treating the received data needs to consider the nature of perturbing field in order to make accurate measurements such as powers of punctual sources, their locations and the types of waveforms which can be deterministic or random. In this paper, we study the type of temporally white and spatially coherent noise field; we propose a new spatial coherence function using Lorentz function. After briefly describing some existing models, we numerically study the effect of spatial coherence length on resolving the angular locations of closely radiating sources using spectral techniques which are divided into beam forming and subspace based methods, this study is made comparatively to temporally and spatially white noise with the same power as the proposed one in order to make precise comparisons.

Keywords: Spatial coherence function, narrowband, direction of arrival, Lorentz function, coherence length.

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1. Introduction

Among the applications of narrowband radio communications [4, 11, 14], is the treatment of interactions between waves which results in spatial interferences that are related to the notion of coherence [4, 15, 16]. The effect of coherence can be spatial, temporal or both. In fact, given a wave field \( E(r, t) \) at instant \( t \) and position \( r \), the coherence gives the period and the range where we can predict the characteristics of the field \( E(r', t') \). In the majority of applications such as radio location and underwater acoustics [18], the coherence of the total electric field and that of perturbing field becomes a problem, because for these applications, the goal is to characterize a wave field by separating the components of each elementary source such as the spatial position \((r, \theta, \phi)\) and power \( \sigma^2 \).

If the total electric field is totally or partially coherent, then some pre-processing treatments are mandatory to decorrelate the sources [6]. In order to explain this process of decorrelation, let us take an example of simple configuration consisting of an array of omnidirectional receiving sensors and two transmitting sources which are closely spaced and temporally coherent, a simple processing of the received data by the array indicates that the bearing is originated from one source, therefore, to make a correct analysis, the decorrelation is necessary which is based on some statistical computing. In other cases, the emitting sources can be temporally uncorrelated, but the perturbing field intercepted by the array can be spatially coherent, the noise field can be localized in the vicinity of the array or made in the far field [10].

The power and the spatial coherence length of this interfering field have an impact on result analysis; they can degrade the resolution power and the interpretation of the obtained results. As the coherence is characterized by temporal and spatial parameters, the four possible cases are: spatial and temporal coherence which is the case for an almost ideal monochromatic wave, spatial incoherence and temporal coherence, spatial and temporal incoherence, finally, the case of temporal incoherence and spatial coherence, in this study we are focused on the last possibility. In terms of statistical description, the third case is said to be temporally and spatially ergodic process in wide sense, it is a realization of white noise [8] described by Gaussian probability distribution. For the fourth possibility, if we make measurements in different positions, we find a pattern of spatial coherence which can be total or partial (degree of correlation). In the literature, the spatial coherence functions have, in general, the same property, indeed, if we measure the powers between several points \( r_i \), we find that the pattern decreases [5, 7, 13], and may become zero for large distance \( |r_i - r_j| \).

In the other cases, given that the sensors are designed to function with defined frequency ranges, then the distance between elementary sensors have an impact of the noise field, in fact, the coherence can be stronger if the distance between consecutive sensors is strictly less than half of the central wavelength \( \lambda \) [5] that corresponds to the operational frequency of the array. Based on these observations, many modeling functions have been proposed [3, 17] to study the
effect of coherence on spectral analysis of the radiating sources. Our objective in this study is to propose a new spatial coherence function for an array of identical and omnidirectional sensors operating with the same carrier frequency as that of punctual and far field sources, we model the spatial pattern of coherence by Lorentz function [2] and we discuss some simulation results of this model in terms of spatial accuracy of some high resolution angle of arrival techniques.

This paper is organized as the following, in literature review section, we describe the data model according to uniform linear array in the context of far field radiation and we present a brief description of some models of spatial coherence. In the third section, we propose a new spatial coherence model using Lorentz function. Next, we discuss in the fourth section the results of Monte Carlo simulation for closely sources.

2. Literature Review

We consider a uniform linear array of \( N \) isotropic sensors separated by distance \( d \) and described by position vector \( r = [0, d, \ldots, (N-1)d] \), the far field contributions of \( P \) radiating sources create a wave field that is linearly polarized. At the reception point \( r_j \), the induced voltage by the electric field \( E(r_j,t) \), after down conversion and sampling, can be written as:

\[
x_j(t) = \sum_{i=1}^{P} s_i(t)e^{-j2\pi l/r_j \sin(\theta)} + n_j(t)
\]  

Where \( s_i(t) \) is the \( i^{th} \) slowly varying envelope [11]. \( \lambda \) is the wavelength which is the same for all sources, \( \theta \) is the \( i^{th} \) angle of arrival relatively to the normal of the array. \( n_j(t) \) is the additive noise and \( r = 1,\ldots,K \). Given that \( N \) channels are available, the data is written in vector form as:

\[
x(t) = a(\theta)s(t) + n(t)
\]  

\( x(t) \in \mathbb{R}^{N \times 1} \) and \( a(\theta) \in \mathbb{R}^{N \times P} \) is the steering matrix [11] that depends on the geometry of the array where we have \( a(\theta) = [a_1(\theta),\ldots,a_p(\theta)] \), the \( i^{th} \) steering vector is written as:

\[
a_i = [1,e^{-j2\pi l/r_i \sin(\theta)},\ldots,e^{-j2\pi l/(N-1)d(\sin(\theta)/2)}]^T
\]  

\( s(t) \in \mathbb{R}^{P \times 1} \) is the envelope vector and \( n(t) \in \mathbb{R}^{N \times 1} \) is the noise vector which is generally described by zero mean complex random process [8] that verifies the following relations:

\[
\begin{align*}
<n_i(t)n^*_j(t+\tau)> &= \sigma^2 \delta(\tau) \\
<n_i(t)n^*_i(t)> &= \sigma^2 \delta(0) \\
<s_i(t)n^*_j(t)> &= 0 \\
<n(t)n^*(t)> &= \sigma^2 I_N
\end{align*}
\]  

Where \( <.> \) is the time average operator, \( \sigma^2 \) is the noise power for \( N \) channels, \( (\cdot)^* \) is the conjugate transpose operator and \( I_N \) is the identity matrix.

Based on the relations of Equation (4), the spatial correlation matrix is defined by:

\[
\Gamma = <xx^*> = a(s(t)s^*(t)) + \sigma^2 I_N = \Gamma_s + \Gamma_n
\]  

The majority of angular spectral techniques are based on \( \Gamma \) to locate the directions of propagation [8, 11], theirs spatial resolutions depend on many factors such as the statistical correlation of noise field. If \( n(t) \) has correlation pattern that depends on array’s geometry, the resolution power of some angle of arrival estimation techniques may be degraded [12]. In the case of spatial coherence, the noise field operator \( \Gamma_n \) is not diagonal but banded matrix, for example in [13], the spatial correlation is described by the operator:

\[
\Gamma_n(u,v) = \sigma^2 \rho^{u-v}
\]  

For parameter \( 0 \leq \rho \leq 1 \), we can remark that in the case \( \rho = 0 \), \( \Gamma_n \) corresponds to white noise matrix.

Another model of spatially correlated noise is given by the operator [1]:

\[
\Gamma_s(u,v) = \begin{cases} 
\sigma^2 \rho^{||u-v||} e^{j(\pi u v/2)} & \text{if } ||u-v|| \leq l \\
0 & \text{else}
\end{cases}
\]  

Where \( l \) is the spatial correlation length \( l < N \), we remark that if \( l = 0 \) then \( \Gamma_s \) is also diagonal matrix.

Another well known model for spatially correlated noise field is the spherically isotropic model [5, 17], where the field received by the array is coming from sources uniformly distributed on the surface of a sphere surrounding the array where the radius is much larger than the array’s length \( D = (N-1)d \). The noise operator is given by the following expression:

\[
\Gamma_s(u,v) = \sigma^2 \frac{\sin(k|u-v|)}{|k|u-v|}
\]  

Where \( k \) is the wave number \( k = 2\pi / \lambda \), this model is used for distances less than half of the wavelength, because we can remark that for an \( N \) uniform linear array of \( d = \lambda / 2 \), the operator describes the case of spatially uncorrelated noise \( \sin(knd)/knd) = 0 \) for \( n=1,\ldots,N-1 \). Based on these models, we present in the next section a new spatial correlation pattern using Lorentz function.

3. Proposed Spatial Coherence Function

In this section, we present a new model of the spatial coherence of noise field which is a function of noise power and spatial coherence length \( l \), the model is based on the following assumptions:
The field is characterized by uniform power for all measurement points $\sigma^2$.

The field is temporally uncorrelated, thus we have the property $<n_i(t)n_j^*(t+\tau)> = \sigma^2 \delta(\tau)$.

Given $\sigma^2$ for the $i^{th}$ sensor, the field is said to be uncorrelated at the $j^{th}$ sensor if the interacting power does not exceed 10% of $\sigma^2$, therefore the spatial coherence length is defined as $l = |r_j - r_i|$.

Given the noise field $n(t) \in \mathbb{C}^{N \times K}$, the real and imaginary parts follow the same model.

The spatial coherence function is version of Lorentz function [2], for distance $|r_j - r_i|$ we have:

$$f(r_j, r_i, \sigma^2, \beta) = \frac{\sigma^2}{\beta(r_j - r_i)^2 + 1}$$

(9)

Where $\beta$ is a parameter that controls the width which must be a function of $l$. If we measure the power, for linear array, at point $r_i$ where $f(r_i, r_i, \sigma^2, \beta) = \sigma^2$, the correlation at position $r_j$ is negligible if $f < 0.1 \sigma^2$ such as the spatial correlation length is $l = |r_j - r_i|$, from this assumption, the width parameter is given by the following criterion:

$$\beta = \frac{9}{l}$$

(10)

In this case, the full width at half maximum equals $FWHM = 2/\sqrt{\beta}$. To study the coherence as function of distance, we compare the proposed function with spherically isotropic noise model and exponential model as represented in Figure 1.

![Figure 1](image_url)

Figure 1. Comparing three different models of spatially coherent noise field.

The parameters of the proposed model are $d = \frac{0.4 \lambda}{l} = \frac{6d}{\lambda}$ and $\sigma^2 = 1 \text{ W}$, for the exponential model, the parameters are $\rho = 0.5$ and $l = 6d$. We remark that the Lorentz model is an envelope approximation of the spherical model. After characterizing the function, we present the computational method to generate the complex data $n(t)$, received by the array, which is temporally white and follow the spatial Lorentz function. The inputs are the number of sensors $N$, the distance as function of wavelength $d(\lambda)$, the spatial coherence length $l$, the uniform power $\sigma^2$ and the number of digital samples $K$. The outputs are the realization of the noise field matrix $n(t)$, the theoretical and estimated correlation matrices $\Gamma_{th}$ and $\Gamma_n$.

- Inputs $N, l, d = f(\lambda), \sigma^2, K, \beta$.
- For $u=1,...,N$ and $v=1,...,N$ compute:
  $$\Gamma_{th}(u,v) = \frac{\sigma^2}{\beta(d(u-v))^2 + 1}$$

- Generate matrix $n_0(t) = \frac{1}{\sqrt{2}}(a(t) + jb(t))$ where $a(t), b(t) \in N(0, N(I_N))$ and $t=1,...,K$.
- Generate matrix $n'(t) = \Gamma_{th}^{1/2} n_0(t)$.
- Estimate correlation matrix $\Gamma_n = n'(t)n''(t)/K$.

We remark that the theoretical correlation matrix is real $\Gamma_{th} \in \mathbb{R}^{N \times N}$ while the estimated matrix is complex valued $\Gamma_n \in \mathbb{C}^{N \times N}$, both are hermitian $\Gamma_{th}^* = \Gamma_{th}, \Gamma_n^* = \Gamma_n$. The matrix $\Gamma_n$ is accurately estimated if the number of samples $K$ is sufficiently large. The quality of estimation can be verified by computing the Root Mean Square Error of eigenvalues as:

$$RMSE = \sqrt{\frac{1}{N} \sum_{j=1}^{N}(\lambda_{th,j} - \lambda_{nj})^2}$$

(11)

Where $\lambda_{th,j}$ are the eigenvalues of $\Gamma_{th}$ and $\lambda_{nj}$ are those of $\Gamma_n$. As previously discussed for other models of spatially coherent noise, this model converges to spatially uncorrelated noise if the spatial coherence length $l$ tends to zero:

$$\lim_{l \to 0} \Gamma_n = \sigma^2 I_N$$

(12)

After generating the new matrix $n'(t)$, the signal model of the array becomes $x'(t) = a(\theta)s(t) + n'(t)$ where the parameter $l$ has in impact on the accuracy of estimating the directions of the $P$ propagating waves using spectral techniques [12], this relationship is discussed in the next section.

4. Results and Discussion

Given the Lorentz model of spatially coherence noise field, we run in this part, some computer simulations to verify the proposed formalism. We verify the resolution power of direction of arrival techniques [8, 11] for specific value of spatial correlation length $l$. 
We consider an array of \( N=18 \) isotropic sensors where the gain of each sensor is \( g(\theta)=1 \). The array is operating with one carrier frequency \( f=c/\lambda \) and the distance between sensors is set to \( d=0.4\lambda \). We simulate the spatially coherent noise field \( n'(t) \) using the steps described in the previous section with parameters \( l=12d \), \( K=300 \) and varying \( \sigma^2 \). We begin the analysis of radiating sources where we consider the position of the array as the origin of the referential. We assume the presence of \( P=3 \) far field and punctual sources that are producing radiations with approximately the same frequency of the array where the type of waveforms is modeled by \( s(t) \sim \mathcal{CN}(0_{P\times 1},I_P) \). The locations of the sources correspond to angles of arrival \( \theta=[21^\circ,24^\circ,29^\circ] \) which we use to generate the steering matrix \( a(\theta) \) and the signal model \( x(t) \) in the presence of spatially coherent noise field. Given the array length \( D=(N-1)d=6.8\lambda \), Rayleigh limit of angular resolution of this array is approximately equal to \( \theta_{\text{HPBW}} \approx \lambda/dN \approx 8^\circ \), therefore estimating the angles of arrival requires high resolution methods, before discussing these spectral techniques, we study in the first part, the estimation of \( \Gamma_n \) in the case of spatially correlated noise from data \( x(t) \).

The spatial correlation matrix can be decomposed into the form \( \Gamma=UU^* \) where \( U \in \mathbb{C}^{N\times N} \) is unitary matrix and \( \Lambda \in \mathbb{C}^{N\times N} \) is diagonal matrix where the diagonal elements are the eigenvalues given in the follow order \( \{\lambda_1,...,\lambda_p,\lambda_{p+1},...,\lambda_N\} \) [14].

The first \( P \) eigenvalues have higher magnitudes and correspond to the signal subspace \( U_s \in \mathbb{C}^{N\times P} \), the remaining \( N-P \) eigenvalues approximately equal the noise power and belong to the noise subspace \( U_n \in \mathbb{C}^{N\times N-P} \) [14] such as \( U=[U_s,U_n] \).

In the case of temporally and spatially ergodic noise, the projector into the noise subspace \( P_n=U_nU_n^* \) is orthogonal to the steering matrix \( a(\theta) \):

\[
P_n a(\theta) = 0_{N\times P} \tag{13}
\]

From this property, it is possible to estimate the operator \( \Gamma_n \) using the following equations:

\[
\begin{align*}
F &= P_n x(t) \approx P_n n(t) \\
\hat{n}(t) &= P_n^* F \\
\hat{\Gamma}_n &= \hat{n}(t)\hat{n}^*(t)/K
\end{align*}
\tag{14}
\]

Where \( (\cdot)^* \) is the generalized inverse operator, using these steps, we compare the estimation of ergodic noise \( n(t) \) and coherence model \( n'(t) \) with respect to power while the powers of the source are kept at \( \sigma_t^2 \approx 1 \) W, the Signal to Noise Ratio is defined by the relation \( \text{SNR}=10\log_{10}(1/\sigma_t^2) \) which varies from -10 dB to 20 dB where, for \( L=20 \) trials of each value of \( \text{SNR} \), we compute the estimate \( \hat{\Gamma}_n \) for both noise models using Equations (14). We measure the efficiency of estimation by average power of the error defined by the relation:

\[
<e>=Tr(\hat{\Gamma}_n - \Gamma_n)/N
\tag{15}
\]

Where \( \text{Tr}(\cdot) \) is the trace operator which is the sum of diagonal elements, the results of the estimation are given in Figure 2.

For level of \( \text{SNR}<10 \) dB, the average error of spatially coherence noise field with \( l=12d \) is higher than that of white noise, staring from 10 dB, estimating the noise coherence matrix \( \Gamma_n \) is more accurate. From this result we fix the noise level at \( \text{SNR}=10 \) dB, which corresponds to noise power \( \sigma^2 = 0.1 \) W, for the second simulation which consists of estimating the angles of incidences \( \theta_i \) of radiating sources.

We fix the number of Monte Carlo trials at \( L=200 \), the angular region of scan is \( \Omega=[15^\circ,40^\circ] \) with angular step \( d\theta=0.1^\circ \). For each value of \( \theta \) we generate the steering vector \( a \in \mathbb{C}^{N\times 1} \) by which the localization function [8] is given by:

\[
f(\theta \in \Omega) = \frac{1}{a^* P_n a}
\tag{16}
\]

Where \( P_n \in \mathbb{C}^{N\times N} \) is chosen spectral operator. We compare the resolution power of average localization function, over \( L \) trials, for Minimal Variance Distortionless Response operator (MVDR) [19] as shown in Figure 3.

We remark that the MVDR technique in the case of spatially coherent noise field has better resolution ability. Next, we compare the Multiple Signal Classification technique (MUSIC) [14] in Figure 4 where the localization function is not successful in the case coherent noise field because the closely sources
are not separated.

Figure 3. MVDR localization function, SNR=10 dB, l=12d.

Finally, we compare the resolution power of the lorentzian operator, a recently proposed technique based on beamforming optimization [9], the result is given in Figure 5.

Figure 4. MUSIC localization function, SNR=10 dB, l=12d.

Figure 5. Lorentz localization function, SNR=10 dB, l=12d.

We conclude from these simulation results that in moderate SNR conditions (10 dB) and for large number of sensors (N=18), the beam forming based techniques have a better resolution power in the case of spatially coherent noise field, comparatively to the subspace based techniques when the spatial coherence length l is almost 70% of array’s length D.

4. Conclusions
In this paper, we have proposed a new spatial coherence function of noise field received by an array of sensors in the context of narrowband and far field source localization problem where the radiating sources are stationary over the period of observation. In the first part, we have briefly described some existing models of spatially correlated noise, next we have proposed a new model based on Lorentz function. In the second part, we conducted some computer simulations to test the robustness of angle of arrival estimation techniques in the presence of such field, the results proved that for large array, closely sources, moderate perturbation level and 70% of spatial correlation length with respect to array length, the beam forming techniques have a better resolution power compared to subspace based techniques.

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Appendix

In this part, we present the Mathworks code to generate the spatially correlated noise field, the inputs are the number of antenna elements N, the distance between the array elements d, the noise standard deviation s, the spatial coherence length l and the number of realizations K. The outputs are the theoretical spatial correlation matrix f, the matrix of noise field n2 and the estimated spatial correlation matrix R.

ULA=(0:N-1)*d;
r=toeplitz(ULA);
p=s^2;
b=9/(l*d)^2;
f=(p)./((b*r.^2)+1);
n=(1/sqrt(2))*(randn(N,K)+j*randn(N,K));
n2=f^(1/2)*n;
R=n2*n2'/K;